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NOTES ON MATRIX THEORY—X  
A PROBLEM IN CONTROL

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Summary

*the integral of  $(x, Bx)dt$  between  
0 and infinity*

→ In the theory of control processes, it is important to be able to calculate  $\int_0^{\infty} (x, Bx)dt$  without having to solve explicitly the differential equation  $dx/dt = Ax$ ,  $x(0) = c$ . A method for doing this is presented in this paper, generalizing one due to Anke for  $n^{\text{th}}$  order linear differential equation. ( ) ↑

# NOTES ON MATRIX THEORY—X

## A PROBLEM IN CONTROL

By

Richard Bellman

### §1. Introduction

In a recent paper, [1], Anke showed that the expression

$$(1) \quad J = \int_0^{\infty} x^2 dt$$

could be computed as a rational function of the coefficients,  $a_1, a_2, \dots, a_n$  in the differential equation for  $x$ ,

$$(2) \quad \frac{d^{(n)}x}{dt^n} = a_1 \frac{d^{(n-1)}x}{dt^{n-1}} + \dots + a_n x,$$

and the initial values  $x(0) = c_1, x'(0) = c_2, \dots, x^{(n-1)}(0) = c_{n-1}$ ,

without solving the equation explicitly, provided that all the solutions of (2) approached zero as  $t \rightarrow \infty$ . This is equivalent to the condition that all the roots of the equation

$$(3) \quad r^n + a_1 r^{n-1} + \dots + a_n = 0$$

have negative real parts. Determinantal criteria for this were first given by Hurwitz.

In this paper we wish to consider the more general problem of determining the value of

$$(4) \quad J = \int_0^{\infty} (x, Bx) dt$$

under the assumption that  $x$  is the solution of

$$(5) \quad \frac{dx}{dt} = Ax, \quad x(0) = c,$$

where the characteristic roots of  $A$  have negative real parts;  
i.e., a stability matrix.

## §2. The Analytic Procedure

Let  $x$  be the solution of (1.5) and compute, for  $F$  a constant symmetric matrix,

$$\begin{aligned} (1) \quad \frac{d}{dt} (x, Fx) &= \left( \frac{dx}{dt}, Fx \right) + \left( x, F \frac{dx}{dt} \right) \\ &= (Ax, Fx) + (x, FAx) \\ &= (x, (A'F + FA)x). \end{aligned}$$

From this it follows that

$$(2) \quad \int_0^{\infty} \frac{d}{dt} (x, Fx) dt = \int_0^{\infty} (x, (A'F + FA)x) dt$$

or

$$-(c, Fc) = \int_0^{\infty} (x, (A'F + FA)x) dt,$$

since, by assumption  $x(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

Hence if  $F$  is determined by the relation

$$(3) \quad A'F + FA = B,$$

we have a solution to the problem posed above.

### §3. The Matrix Problem

The question arises as to the existence of a solution of the equation in (2.3). Fortunately the problem can be resolved very simply by means of a transcendental procedure. It is well-known that one solution of this equation is

$$(1) \quad P = -\int_0^{\infty} e^{A't} B e^{At} dt,$$

well-defined for A a stability matrix.

Since this solution exists for arbitrary B, it follows that it is unique. Hence (2.3) can always be solved by the standard determinantal method.

It is interesting to note that in the (2 x 2) case, the determinant of the 3 unknown elements in the symmetric matrix P is the product of tr A by det A. The conditions that A be a stability matrix are that  $\text{tr } A < 0$ ,  $\det A > 0$ .

It is tempting to conjecture that the factors of the determinant of the  $N(N + 1)/2$  unknown elements in the symmetric matrix P in the general case constitute a set of Hurwitz criteria for the matrix. In the case  $N = 3$ , the determinant is of degree 6. This leaves room for a quadratic factor in addition to the linear factor, tr A, and the cubic factor det A.

If a systematic method of obtaining these factors existed, the problem of determining stability criteria directly in terms of the elements of A, rather than in terms of the coefficients of the characteristic polynomial of A, would be resolved.

References

1. Klaus Anke, "Eine Neue Berechnungsmethode der quadratischen Regelfläche," Zeit. ang. Math. and Physik, Vol. VI (1955), pp. 327-331.

see also

2. H. Bückner, "A Formula for an Integral Occurring in the Theory of Linear Servomechanisms and Control Systems", Quart. Appl. Math., 10 (1952).
3. P. Hazebroek and B. I. Van der Waerden, "Theoretical Considerations in the Optimum Adjustment of Regulators", Trans. Amer. Soc. Mech. Engineers, 72 (1950).